Technical Comments

Comment on "Stochastic Stability of a Satellite Influenced by Aerodynamic and Gravity Gradient Torques"

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THE region of stochastic stability of a pitching satellite determined by Sheporaitis[1] can be considerably enlarged. Consider the case $g(x_1) = \sin x_1$. Linearizing the pitch equation[1] and using the author's abbreviations we get the linear Itô-system

$$dx = Axdt + Bxdz \tag{1}$$

with

$$A = \begin{bmatrix} -Q\eta & 1 \\ -Q(1-2l) & 0 \end{bmatrix} \qquad B = -Q\delta \cdot \begin{bmatrix} 0 & 0 \\ 1 - Q\eta^2 & \eta \end{bmatrix}$$

The covariance matrix $P = Exx^T$ satisfies the covariance equation $\dot{P} = AP + PA^T + BPB^T$ and can be shown to be stable for

$$\delta_0^2 < 2\eta_0(1-2l)/[1 + (1-2l)\eta_0^2] \tag{2}$$

i.e., the solution of the linearized equation is stable in mean square for δ_0^2 satisfying Eq. (2). The mean square stability implies the stability with probability 1. The stochastic stability of the linearized equation implies the stability of the original nonlinear equation. Thus, the stability condition for the pitch motion is Eq. (2). The parameter region corresponding to condition (2) is considerably larger than the region of Fig. 2 obtained in Ref. 1 by Liapunov techniques.

However, the condition (2) can be derived by Liapunov techniques too if the constants of the Liapunov function are properly chosen. Set $a = 1$, $c = b/2$, $d = 2a(1 - Q\eta\delta^2)$ as in Ref. 1 but do not fix the constant b. Then, supposing $b < 2$ the inequality

$$-Lv \geq Q\eta(b - Q\eta\delta^2)(x_2 - Q\eta x_1)^2 + Q^2 x_1 \sin x_1 [2\eta - b\eta - 2\eta l(2-b) - \delta^2]$$

holds and, in consequence, Lv is negative definite for

$$\delta^2 < \min\{b/Q\eta, 2\eta - b\eta - 4\eta l + 2b\eta l\} \tag{3}$$

We get the widest range for δ^2 or $\delta_0^2 = \delta^2(Q)^{1/2}$ if

$$b/Q\eta = 2\eta - b\eta - 4\eta l + 2b\eta l$$

i.e., for

$$b = 2Q\eta^2(1-2l)/[1 + Q\eta^2(1-2l)]$$

For this value of b condition (3) is identical with (2). It remains to show that for δ_0^2 satisfying (2), the Liapunov function v is positive definite. Indeed, regarding (2) the function v can be minimized by

$$v \geq \left(x_2 - \frac{b}{2} Q\eta x_1\right)^2 + \frac{4Q(1-2l)}{1 + Q\eta^2(1-2l)} \left(\frac{x_1}{2}\right)^2 \times \left[\frac{Q^2\eta^4}{1 + Q\eta^2(1-2l)} + (1 - (1+2l)Q\eta^2)\left(\frac{\sin x_1/2}{x_1/2}\right)^2\right]$$

If $1 - (1+2l)Q\eta^2 > 0$ the right-hand side is positive and thus $v \geq 0$. If $1 - (1+2l)Q\eta^2 < 0$ the term in square brackets is greater than

$$\frac{Q^2\eta^4}{1 + Q\eta^2(1-2l)} - (1+2l)Q\eta^2 + 1 = \frac{(2lQ\eta^2 - 1)^2}{1 + Q\eta^2(1-2l)} > 0$$

and, consequently, $v \geq 0$ holds in this case too, i.e., condition (2) is sufficient for the stability with probability one, Q.E.D. A further enlargement of the stability domain by means of the Liapunov function given[1] seems to be impossible.

Reference

[1] Sheporaitis, L. P., "Stochastic Stability of a Satellite Influenced by Aerodynamic and Gravity Gradient Torques," *AIAA Journal*, Vol. 9, No. 2, Feb. 1971, pp. 218–222.

Received June 1, 1971.
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Reply by Author to P. S. Sagirow

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THE author would like to thank S. Sagirow for his observation concerning the regions of stochastic stability for the pitching satellite. The inequality shown (ii) essentially doubles the region for stability with probability one. It also points the way towards expanding the regions in the normalized phase space (Fig. 3 of Ref. 1) that guarantee convergence to the origin with a given probability. However, in this case the constant b must be chosen as

$$b = 2Q\eta^2(1+2l)/[1 + Q\eta^2(1+2l)] \tag{1}$$

The reason for this is that the latter stability criterion is a global one and x_1 cannot be considered to be restricted to $[0, \pi/2]$. This follows from

$$\mathcal{L}v(x) = Q\eta(-b + Q\eta\delta^2)(Q\eta x_1 - x_2)^2 + x_1 \sin x_1 Q^2\eta(b-2)(1 - 2l\cos x_1) + Q^2\delta^2 \sin^2 x_1$$

when x_1 takes values in $[0, \pi]$

$$-\mathcal{L}v(x) \geq Q\eta(b - Q\eta\delta^2)(x_2 - Q\eta x_1)^2 + Q^2 x_1 \sin x_1 [2\eta - b\eta + 2\eta l(2-b) - \delta^2]$$

This gives rise to the value of b in Eq. (1) to insure negative definiteness. The lower bound on $v(x)$ shown by S. Sagirow modified with this constant b could then be used to find ellipses within which convergence to the origin were guaranteed with a given probability.

Reference

[1] Sheporaitis, L. P., "Stochastic Stability of a Satellite Influenced by Aerodynamic and Gravity Gradient Torques," *AIAA Journal*, Vol. 9, No. 2, Feb. 1971, pp. 218–222.

Received July 19, 1971.
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